Turán's Theorem Formalization

ro-gut

September 24, 2025

0.1 Concentrating support on a clique - Improve Operartion

Definition 1 (Weight distributions). Given a finite simple graph G = (V, E), a weight distribution is a function

$$w: V \to \mathbb{R}_{\geq 0}$$
 with $\sum_{v \in V} w(v) = 1$.

A probability distribution on the vertex Set.

Definition 2 (Total edge weight). For $w \in W$, the total edge weight is

$$f(w) := \sum_{e \in E} \operatorname{vp}(w, e).$$

In words: sum the edge values over all edges of the graph.

Definition 3 (A better distribution). Given a weight function w, define w^* such that $\operatorname{supp}(w^*) \subseteq \operatorname{supp}(w)$ and

$$w.\text{fw} < w^*.\text{fw}$$
.

That is w^* is a distribution with non-decreasing total edge weight with the original support of w preserved.

Definition 4 (Improve Operation). Given distinct vertices $v_j \neq v_i$, define $w' := \text{Improve}(W, v_j, v_i)$ by moving all weight from v_i to v_i :

$$w'(v_j)=0, \qquad w'(v_i)=w(v_i)+w(v_j), \qquad w'(v)=w(v) \text{ for } v \notin \{v_i,v_j\}.$$

Lemma 5 (Sum split). The sum over edges E of the function vp splits as

$$\sum_{e \in E} \mathit{vp}(w', e) = \sum_{\substack{e \in E \\ v_i \in e}} \mathit{vp}(w', e) + \sum_{\substack{e \in E \\ v_j \in e}} \mathit{vp}(w', e) + \sum_{\substack{e \in E \\ v_i, v_j \notin e}} \mathit{vp}(w', e).$$

That is the total sum decomposes into parts incident to v_i , incident to v_i , and the rest.

Lemma 6 (Gain-incidence increases). The increment in the sum over edges incident to v_i is

$$\sum_{v_i \in e} \mathit{vp}(w', e) - \sum_{v_i \in e} \mathit{vp}(w, e) = \varepsilon \sum_{v_y \in N(v_i)} w(v_y),$$

where $N(v_i)$ is the neighborhood of v_i .

Lemma 7 (Loose-incidence becomes zero). The sum over edges incident to v_j after the transfer satisfies

$$\sum_{v_j \in e} \mathit{vp}(w', e) = 0.$$

Lemma 8 (Unchanged complement). For edges e not incident to v_i or v_j ,

$$vp(w',e) = vp(w,e).$$

Edges outside v_i and v_j neighborhoods remain unchanged under the operation.

Lemma 9 (Improve results in non decreasing distribution). Given non adjacent vertices v_i , v_j and assuming that the neighborsum of weights at s_i is equal or greater than that of s_j , we have

$$f(w) \le f(w')$$
.

that is the total edge weight does not decrease after applying Improve.

Lemma 10 (Improve strictly reduces support). The new distribution w' has a strictly smaller support:

$$|\operatorname{supp}(w')| < |\operatorname{supp}(w)|.$$

Theorem 11 (Support of Better is a clique). The support of w^* forms a clique:

$$\forall v_x, v_y \in \operatorname{supp}(w^*), v_y \neq v_x \implies \{v_y, v_x\} \in E.$$

In words: every two distinct vertices with positive weight in w^* are adjacent.

0.2 The Enhance Operation

Definition 12 (Enhance Operation). Given distinct non-adjacent vertices v_j, v_i , define w^+ by transferring $\varepsilon > 0$ weight from v_i to v_i :

$$w^{+}(v_{i}) = w(v_{i}) - \varepsilon$$
, $w^{+}(v_{i}) = W(v_{i}) + \varepsilon$, $w^{+}(v) = W(v)$ for $v \neq v_{i}, v_{i}$,

with the condition $\{v_j, v_i\} \notin E$. The Enhance transfers weight between non-adjacent vertices to increase edge weight.

Lemma 13 (Supported edge partition). The edge set E partitions as

$$E = E_{v_i} \cup E_{v_i} \cup E_{\text{rest}},$$

where

$$E_{v_i} = \{e \in E : v_i \in e\}, \quad E_{v_j} = \{e \in E : v_j \in e\}, \quad E_{\mathrm{rest}} = E \smallsetminus (E_{v_i} \cup E_{v_j}).$$

In words: edges are split into the incidence set to v_i , the one to v_i , and the rest.

Lemma 14 (Enhance gain sum). Under Theorem 12, the change in the sum over edges incident to v_i satisfies

$$\sum_{e \in E_{v_i}} \mathit{vp}(w^+, e) - \sum_{e \in E_{v_i}} \mathit{vp}(w, e) = \varepsilon \sum_{v_y \in N(v_i)} w(v_y).$$

That is the gain vertex's edge contribution increases by ε times the sum of its neighbors' weights.

Lemma 15 (Enhance loose sum). Under Theorem 12, the sum over edges incident to v_i satisfies

$$\sum_{e \in E_{v_j}} \mathit{vp}(w^+, e) = 0.$$

That is the loose vertex's incident edge contributions become zero after Enhance.

Definition 16 (Bijection inside the clique). Define a bijection

$$\phi:\{e\in E_{v_i} \smallsetminus \{s(v_j,v_i)\}\} \to \{e\in E_{v_i} \smallsetminus \{s(v_j,v_i)\}\}$$

mapping edges incident to v_j (except $s(v_j, v_i)$) to edges incident to v_i (except $s(v_j, v_i)$). In words: this bijection pairs edges incident to v_i with edges incident to v_i within the clique.

Lemma 17 (Bijection preserves). For any edge e incident to v_j (excluding $s(v_j, v_i)$), the "other" vertex weight satisfies

$$w(\text{other}(e, v_i)) = w(\text{other}(\phi(e), v_i)).$$

In words: the bijection preserves weights at the other endpoints of edges.

Lemma 18 (Loose/gain equality). The total weight transfer balances the edge contributions:

$$\sum_{e \in E_{v_j}} \mathit{vp}(w^+, e) + \sum_{e \in E_{v_i}} \mathit{vp}(w', e) \geq \sum_{e \in E_{v_j}} \mathit{vp}(w, e) + \sum_{e \in E_{v_i}} \mathit{vp}(w, e).$$

In words: the combined edge contributions of loose and gain vertices do not decrease after Enhance.

Lemma 19 (Complement unchanged). For edges $e \in E_{rest}$,

$$vp(w^+, e) = vp(w, e).$$

In words: edges not incident to v_i or v_j remain unaffected by Enhance.

Lemma 20 (Edge contribution increase). The total edge contribution satisfies:

$$\sum_{e \in E_{v_i} \cup E_{v_j}} \mathit{vp}(w^+, e) \geq \sum_{e \in E_{v_i} \cup E_{v_j}} \mathit{vp}(w, e).$$

That is, the total contribution from gain and loose vertices does not decrease.

Lemma 21 (Support edges unchanged). For any vertex $v \notin \{v_i, v_j\}$, the edge contributions satisfy

$$\sum_{e\ni v} \mathbf{vp}(w^+,e) = \sum_{e\ni v} \mathbf{vp}(w,e).$$

In words: vertices outside gain and loose retain their edge contributions after Enhance.

Theorem 22 (Enhance increases edge weight). For a given distribution $w \in W$ Applying Enhance (w^+) strictly increases the total edge weight:

$$f(w^+) > f(w)$$

That is, the Enhance operation strictly improves the total edge weight contribution.

0.3 Equalizing the weights on the clique - EnhanceD

Definition 23 (Maximising the number of uniform vertices). For a given distribution w, K is the maximal number of uniform vertices achievable without decreasing the total edge weight

$$K := \max\{N_a(w)\}$$

.

Lemma 24 (Best uniform distribution exists). There exists w_M with $\operatorname{supp}(w_M) \subseteq \operatorname{supp}(w)$, w_M .fw $\geq W$.fw, and with at least m vertices having weight 1/m. In words: a maximiser w_M achieving the maximal uniform vertex count exists.

Definition 25 (UniformBetter). Given $w \in \mathcal{W}$ whose support induces a clique, define

$$w_M := \mathtt{UniformBetter}(w)$$

to be the witness provided by Theorem 24: it preserves the zero set of W, its support is a clique, satisfies $f(w_M) \geq f(w)$, and achieves the maximal number $K = \mathtt{max_uniform_support}(w)$ of vertices with weight 1/k (where $k = |\operatorname{supp}(w)|$).

Definition 26 (Carefully chosen ε). Define

$$\mathsf{the}_{_} \; := \; w_{\max} \; - \; \frac{1}{k}.$$

In words: the is the difference between the largest vertex weight and the average 1/k.

Definition 27 (Enhanced Operation). Let v_{max} and v_{min} be vertices attaining the maximal and minimal weights of w, respectively. Set $\varepsilon := \mathsf{the}_{_}$ and define

$$w^+ \; := \; \mathtt{Enhance}(w, v_{\max}, v_{\min}, \mathsf{the}_\,).$$

In words: w^+ transfers the carefully chosen ε from the heaviest to the lightest vertex.

Lemma 28. For any vertex v with $w(v) = \frac{1}{|\sup p(w)|}$,

$$w^+(v) = w(v).$$

In words: vertices already at uniform weight remain unchanged under Enhanced.

Lemma 29. The weight at the argmax vertex v_i after Enhanced satisfies

$$w^+(v_j) = \frac{1}{|\mathrm{supp}(w)|}.$$

That is, Enhanced reduces the argmax vertex's weight to the uniform weight.

Lemma 30. The number of vertices with weight $\frac{1}{|\sup(W)|}$ increases after Enhanced:

$$|\{v: w^+(v) = 1/|\sup(w)|\}| > |\{v: W(v) = 1/|\sup(w)|\}|.$$

Lemma 31 (Uniform weights on the support). For every vertex $v \in \text{supp}(w_M)$,

$$w_M(v) = \frac{1}{|\text{supp}(w_M)|}.$$

That is the weights of all support vertices in UniformBetter are uniform.

Lemma 32 (Edge values under Uniform Better). For any edge $e = \{v_a, v_b\}$ with $v_a, v_b \in \text{supp}(w_M)$,

$$\mathit{vp}(w_M, e) = \left(\frac{1}{|\mathrm{supp}(w_M)|}\right)^2.$$

In words: every supported edge has value equal to the square of the uniform vertex weight.

Lemma 33 (Edge count in a clique). If $|\text{supp}(w_M)| = k$, then

$$|\{e \in E : e \subseteq \operatorname{supp}(w_M)\}| = \frac{k(k-1)}{2}.$$

Lemma 34 (computation). For k > 0,

$$\frac{k(k-1)}{2}\cdot \left(\frac{1}{k}\right)^2 = \frac{1}{2}\left(1-\frac{1}{k}\right).$$

That is the total edge weight for a clique with uniform weights simplifies to $\frac{1}{2}(1-\frac{1}{k})$.

Lemma 35 (Monotonicity of the bound). The function

$$f(k) := \frac{1}{2} \left(1 - \frac{1}{k} \right)$$

is nondecreasing for $k \geq 1$.

Theorem 36 (Final bound inside a clique). If w is supported on a clique of size $k \leq p-1$, then

$$f(w) \leq \frac{1}{2} \left(1 - \frac{1}{p-1}\right).$$

In words: the total edge weight is bounded by the Turán bound for cliques of size less than p.

Definition 37 (Uniform weights over all vertices). Define

$${\tt UnivFun}(G)(v):=\frac{1}{|V|} \quad \forall v \in V.$$

That is, the uniform vertex weight distribution assigns equal weight 1/|V| to every vertex.

Lemma 38 (Total weight under UnivFun). The total edge weight satisfies

$$(\mathit{UnivFun}(G)).\mathrm{fw} = |E| \cdot \left(\frac{1}{|V|} \right)^2.$$

THat is, the total edge weight under uniform vertex weights equals the number of edges times the square of the uniform weight.

Theorem 39 (Turán's Theorem). Let $p \geq 2$ and let G be a p-clique-free graph. Then

$$|E| \leq \frac{1}{2} \left(1 - \frac{1}{p-1}\right) |V|^2.$$